IEEE Standard Definitions of Physical Quantities for Fundamental Frequency and Time Metrology—Random Instabilities

IEEE Standards Coordinating Committee 27

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Abstract: Methods of describing random instabilities of importance to frequency and time metrology are covered. Quantities covered include frequency, amplitude, and phase instabilities; spectral densities of frequency, amplitude, and phase fluctuations; and time-domain deviations of frequency fluctuations. In addition, recommendations are made for the reporting of measurements of frequency, amplitude, and phase instabilities, especially in regard to the recording of experimental parameters, experimental conditions, and calculation techniques.
Keywords: AM noise, amplitude instability, FM noise, frequency domain, frequency instability, frequency metrology, frequency modulation, noise, phase instability, phase modulation, phase noise, PM noise, time domain, time metrology

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Introduction

This introduction is not part of IEEE Std 1139-2008, IEEE Standard Definitions of Physical Quantities for Fundamental Frequency and Time Metrology—Random Instabilities.

Techniques to characterize and to measure the frequency, phase, and amplitude instabilities in frequency and time devices and in received radio signals are of fundamental importance to all manufacturers and users of frequency and time technology.

In 1964, Standards Coordinating Committee 14 and, in 1966, the Technical Committee on Frequency and Time were formed to prepare an IEEE standard on frequency stability. In 1969, these committees completed a document proposing definitions for measures of frequency and phase stabilities (Barnes [B14]).[a] In 1988, an updated IEEE standard on frequency stability was published as IEEE Std 1139-1988, IEEE Standard Definitions of Physical Quantities for Fundamental Frequency and Time Metrology. Later on a revision of this standard was published as IEEE Std 1139-1999, IEEE Standard Definitions of Physical Quantities for Fundamental Frequency and Time Metrology—Random Instabilities. The recommended measures of instabilities in frequency generators have gained acceptance among frequency and time users throughout the world.

This standard is a revision of IEEE Std 1139-1999, which had been prepared by a previous Standards Coordinating Committee 27, consisting of John R. Vig, Chair; Eva S. Ferre-Pikal, Vice Chair; James C. Camparo, Leonard S. Cutler, Lute Maleki, William J. Riley, Samuel R. Stein, Claudine Thomas, Fred L. Walls, and Joseph D. White. Some clauses of the 1999 standard remain unchanged.

Most of the major manufacturers now specify instability characteristics of their standards in terms of the recommended measures. This standard thus defines and formalizes the general practice.

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[a] The numbers in brackets correspond to those of the bibliography in Annex F.

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At the time this standard was submitted to the IEEE-SA Standards Board for approval, the 1139 Working Group had the following membership:

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IEEE Standard Definitions of Physical Quantities for Fundamental Frequency and Time Metrology—Random Instabilities

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1. Scope

This standard covers the fundamental metrology for describing random instabilities of importance to frequency and time metrology. Quantities covered include frequency, amplitude, and phase instabilities; spectral densities of frequency, amplitude, and phase fluctuations; and time-domain deviations of frequency fluctuations. In addition, recommendations are made for the reporting of measurements of frequency, amplitude, and phase instabilities, especially in regard to the recording of experimental parameters, experimental conditions, and calculation techniques. The annexes cover basic concepts and definitions, time prediction, and confidence limits when estimating deviations and spectral densities from a finite data set. The annexes also cover translation between the frequency domain and time domain instability measures, examples on how to calculate the time-domain measures of frequency fluctuations, and an extensive bibliography of the relevant literature. Systematic instabilities, such as environmental effects and aging, are discussed in IEEE Std 1193™-2003 [B54].[1]

[1] The numbers in brackets correspond to those of the bibliography in Annex F.

2. Definitions

For the purposes of this standard, the following terms and definitions apply. *The Authoritative Dictionary of IEEE Standards Terms* [B51] should be referenced for terms not defined in this clause. If ambiguities are created between the narrative definition given here and the mathematical equation given in the text, the equation has priority.

2.1 amplitude spectrum $S_a(f)$: One-sided spectral density of the normalized amplitude fluctuations, as defined in **normalized amplitude fluctuations $a(t)$**.

2.2 bias: The difference between the expected value of an estimator of a statistic and its correct value.

2.3 confidence interval: An interval of uncertainty associated with an estimate of an instability measure from a finite number of measurements. The endpoints of a confidence interval are called *confidence limits.*

2.4 frequency spectrum $S_y(f)$: One-sided spectral density of the normalized frequency fluctuations, as defined in **normalized frequency fluctuations $y(t)$**.

2.5 normalized amplitude fluctuations $a(t)$: Instantaneous normalized amplitude departure from a nominal amplitude.

2.6 normalized frequency fluctuations $y(t)$: Instantaneous, normalized frequency departure from a nominal frequency.

2.7 phase fluctuations $\phi(t)$: Instantaneous phase departure from a nominal phase.

2.8 phase noise $\mathcal{L}(f)$: One-half of the phase spectrum $S_\phi(f)$, as defined in **phase spectrum $S_\phi(f)$**.

2.9 phase spectrum $S_\phi(f)$: One-sided spectral density of the phase fluctuations.

2.10 time fluctuations $x(t)$: Instantaneous time departure from a nominal time.

2.11 time interval error (*TIE*): The time difference between a real clock and an ideal uniform time scale following a time period t after perfect synchronization.

2.12 time spectrum $S_x(f)$: One-sided spectral density of the time fluctuations.

2.13 two-sample deviation $\sigma_y(\tau)$: Also called the *Allan deviation*; the square root of the two sample variance, as defined in **two-sample variance $\sigma_y^2(\tau)$**.

2.14 two-sample variance $\sigma_y^2(\tau)$: Also called the *Allan variance;* one-half the time average of the square of the difference between the averages of normalized frequency fluctuations over two adjacent time intervals of length τ, with no dead time between the two averaging intervals.

3. Standards for characterizing or reporting measurements of frequency, amplitude, and phase instabilities

The standard measure for characterizing frequency and phase instabilities in the frequency domain is $\mathcal{L}(f)$, (pronounced "script-ell of f"), defined as one half of the one-sided spectral density of phase fluctuations:

$$\mathcal{L}(f) = \frac{1}{2} S_\phi(f)$$

When expressed in decibels, the units of $\mathcal{L}(f)$ are dBc/Hz (dB below the carrier in a 1 Hz bandwidth). A device shall be characterized by a plot of $\mathcal{L}(f)$ versus Fourier frequency f. In some applications, providing $\mathcal{L}(f)$ versus discrete values of Fourier frequency is sufficient. (See Annex A and Annex B for further discussion). The standard measure for characterizing amplitude instability in the frequency domain is one half of the one-sided spectral density of the normalized amplitude fluctuations, $\frac{1}{2} S_a(f)$. When expressed in decibels the units of $\frac{1}{2} S_a(f)$ are dBc/Hz (dB below the carrier in a 1 Hz bandwidth). See Annex A for a detailed discussion on spectral densities.

In the time domain, the standard measure of frequency and phase instabilities is the Allan deviation $\sigma_y(\tau)$. A device shall be characterized by a plot of $\sigma_y(\tau)$ versus averaging time τ. In some cases, providing discrete values of $\sigma_y(\tau)$ versus τ is sufficient. (See Annex A and Annex B for further discussion.) The measurement system bandwidth (f_h) and the total measurement time shall be indicated.

In addition the provisions in 3.1 and 3.2 are recommended when reporting measurements on frequency and phase instabilities.

3.1 Nonrandom phenomena should be recognized

In particular:

a) Any observed time dependence of the statistical measures should be stated.

b) The method of modeling systematic behavior should be specified (e.g., an estimate of the linear frequency drift was obtained from the coefficients of a linear least-squares regression to M frequency measurements, each with a specified averaging or sample time t and measurement bandwidth f_h).

c) The environmental sensitivities should be stated (for example, the dependence of frequency and/or phase on temperature, magnetic field, barometric pressure, and vibration).

3.2 Relevant measurement or specification parameters should be given

Relevant measurement or specification parameters include the following:

a) The nominal signal frequency ν_o.

b) The method of measurements.

c) The measurement system bandwidth f_h and the corresponding low-pass filter response.

d) The total measurement (data sample) time and the number of measurements N.

e) The characteristics of the reference signal (equal noise or much lower noise assumed).

f) For averaging times which exceed 10% of the total measurement time using $\hat{\sigma}_{y,TOTAL}(\tau)$ to estimate $\sigma_y(\tau)$ is recommended (see Howe [B41], [B42] and Howe and Greenhall [B44]).

g) The calculation techniques [e.g., details of the window function when estimating power spectral densities from time-domain data, or the assumptions about effects of dead time when estimating the two-sample deviation $\sigma_y(\tau)$].

h) The confidence interval (or uncertainty) of the estimate and its statistical probability (e.g., 1σ for 68%, 2σ for 95%). See Annex E.

i) The environment during measurement.

Annex A

(informative)

Measures of frequency, amplitude, and phase instabilities

A.1 Measures of frequency, amplitude, and phase instabilities

The instantaneous output voltage of a precision oscillator can be expressed as

$$v(t) = \left(V_o + \varepsilon(t)\right)\sin\left(2\pi \nu_o t + \phi(t)\right) \tag{A.1}$$

where

- V_o is the nominal peak voltage amplitude
- $\varepsilon(t)$ is the deviation from the nominal amplitude
- ν_o is the nominal frequency
- $\phi(t)$ is the phase deviation from the nominal phase $2\pi \nu_o t$

Figure A.1[2] illustrates a signal with frequency, amplitude, and phase instabilities. As shown, frequency instability is the result of fluctuations in the period of oscillation. Fluctuations in the phase result in instability of the zero crossing. Fluctuations in the peak value of the signal (V_{peak}) result in amplitude instability.

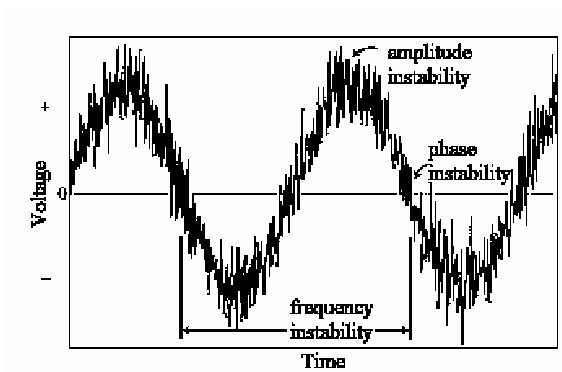

Figure A.1—Instantaneous output voltage of an oscillator

[2] In the signal shown in Figure A.1, the frequency components of the noise are higher than the carrier frequency. The higher frequency components are used for illustration purposes only. In general, this standard applies to the frequency components of amplitude, phase, and frequency instabilities that are lower in frequency than the carrier frequency.

Frequency instability of a precision oscillator is defined in terms of the instantaneous, normalized frequency deviation, $y(t)$, as follows

$$y(t) \equiv \frac{\nu(t) - \nu_o}{\nu_o} = \frac{\dot{\phi}(t)}{2\pi \nu_o} \tag{A.2}$$

where $\nu(t)$ is the instantaneous frequency (time derivative of the phase divided by 2π), and

$$\dot{\phi}(t) = \frac{\mathrm{d}\phi(t)}{\mathrm{d}t} \tag{A.3}$$

Amplitude instability is defined in terms of the instantaneous, normalized amplitude deviation

$$a(t) \equiv \frac{\varepsilon(t)}{V_o} \tag{A.4}$$

Phase instability, defined in terms of the instantaneous phase deviation $\phi(t)$, can also be expressed in units of time, as

$$x(t) = \frac{\phi(t)}{2\pi \nu_o} \tag{A.5}$$

With this definition, the instantaneous, normalized frequency deviation is

$$y(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t} \tag{A.6}$$

Other random phenomena observed in certain oscillators are frequency jumps, that is, discontinuities in the frequency of oscillation. These phenomena are not repetitive or well understood and cannot be characterized by standard statistical methods.

A.2 Frequency domain

In the frequency domain, frequency, amplitude and phase instabilities can be defined or measured by one-sided spectral densities (see Table A.1).

The measure of frequency instability is the spectral density of normalized frequency fluctuations, $S_y(f)$, given by

$$S_y(f) = y_{rms}^2(f) \frac{1}{BW} \tag{A.7}$$

where

$y_{rms}(f)$ is the measured root mean squared (rms) value of normalized frequency fluctuations in a band of Fourier frequencies containing frequency f

BW is the width of the frequency band in Hz

The units of $S_y(f)$ are 1/Hz. This expression for $S_y(f)$ can be derived from the Fourier transform relation between power spectral density and autocorrelation function:

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |y(t)|^2 \, \mathrm{d}t \equiv y_{rms}^2 = \int_0^{\infty} S_y(f) \mathrm{d}f$$

so that for sufficiently narrow Fourier frequency bands (such that $S_y(f)$ is approximately constant over the bandwidth) we have

$$y_{rms}^2 \cong \sum_{k=1}^{\infty} S_y(f_k) BW = \sum_{k=1}^{\infty} y_{rms}^2(f_k)$$

The measure of amplitude instability is the spectral density of normalized amplitude fluctuations, $S_a(f)$, given by

$$S_a(f) = a_{rms}^2(f) \frac{1}{BW} \tag{A.8}$$

The units of $S_a(f)$ are 1/Hz. Again, $a_{rms}(f)$ is to be understood as an rms value in a specific Fourier frequency band of width BW.

Phase instability can be characterized by the spectral density of phase fluctuations, $S_\phi(f)$, given by

$$S_\phi(f) = \phi_{rms}^2(f) \frac{1}{BW} \tag{A.9}$$

The units of $S_\phi(f)$ are rad^2/Hz. Here too, $\phi_{rms}(f)$ is an rms value in a specific Fourier frequency band.

As mentioned previously, $S_y(f)$, $S_a(f)$, and $S_\phi(f)$, are one-sided spectral densities, and apply over a Fourier frequency f range from 0 to ∞. They are equivalent to the sums of the two-sided or single sideband spectral densities (see Table A.1), which are the complex Fourier transforms of their respective autocorrelation functions, for both positive and negative frequencies f and $-f$. This equivalence arises because the complex Fourier transform of a real time-domain process is both real and symmetric in f.

$S_\phi(f)$ is the quantity that has been historically utilized (see Cutler and Searle [B29]) in frequency metrology; however, $\mathcal{L}(f)$ has become the prevailing measure of phase noise among manufacturers and users of frequency standards. According to the *old* definition (see Kartaschoff [B59]), $\mathcal{L}(f)$ is the ratio of the power in one sideband due to phase modulation (PM) by noise (for a 1 Hz bandwidth) to the total signal power (carrier plus sidebands); that is,

$$\mathcal{L}(f) = \frac{\text{power density in one phase noise modulation sideband, per Hz}}{\text{total signal power}} \tag{A.10}$$

Usually $\mathcal{L}(f)$ is expressed in decibels (dB) as 10 $\log_{10}\mathcal{L}(f)$, and its units are dB below the carrier in a 1 Hz bandwidth, generally abbreviated as dBc/Hz.

The *old* definition of $\mathcal{L}(f)$ is related to $S_\phi(f)$ by

$$\mathcal{L}(f) \cong \frac{S_\phi(f)}{2} \tag{A.11}$$

This relationship breaks down when the mean squared phase deviation, $<\phi^2(t)>$ = the integral of $S_\phi(f)$ from f to ∞, exceeds about 0.1 rad^2. To circumvent this difficulty, $\mathcal{L}(f)$ is redefined as

$$\mathcal{L}(f) \equiv \frac{S_\phi(f)}{2} \tag{A.12}$$

This redefinition is intended to avoid difficulties in the use of $\mathcal{L}(f)$ in situations where the small angle approximation is not valid. $\mathcal{L}(f)$, as defined by Equation (A.12), should be designated as the standard measure of phase instability in the frequency domain. The reasons are the following:

— It can always be measured unambiguously, and

— It conforms to the prevailing usage.

Definitions of spectral densities and their relations are given in Table A.1.

Table A.1—Spectral density definitions

Name	Error variance	Relationships (assumes ±*f* symmetry)
One-sided spectral density $S_{1-sided}(f)$	$\int_0^\infty \mathrm{d}f\, S_{1-sided}(f)$	$S_{1-sided}(f) = S_{DSB}(f) = 2S_{2-sided}(f) = 2S_{SSB}(f) = 2\mathcal{L}(f)$
Two-sided spectral density $S_{2-sided}(f)$	$\int_{-\infty}^\infty \mathrm{d}f\, S_{2-sided}(f)$	$S_{2-sided}(f) = S_{SSB}(f) = \mathcal{L}(f) = ½S_{1-sided}(f) = ½S_{DSB}(f)$
Single sideband spectral density $S_{SSB}(f)$	$\int_{-\infty}^\infty \mathrm{d}f\, S_{SSB}(f)$	$S_{SSB}(f) = S_{2-sided}(f) = \mathcal{L}(f) = ½S_{1-sided}(f) = ½S_{DSB}(f)$
Double sideband spectral density $S_{DSB}(f)$	$\int_0^\infty \mathrm{d}f\, S_{DSB}(f)$	$S_{DSB}(f) = S_{1-sided}(f) = 2S_{2-sided}(f) = 2S_{SSB}(f) = 2\mathcal{L}(f)$
"Script ell of f" $\mathcal{L}(f)$	$2\int_0^\infty \mathrm{d}f\, \mathcal{L}(f)$	$\mathcal{L}(f) = S_{SSB}(f) = S_{2-sided}(f) = ½S_{1-sided}(f) = ½S_{DSB}(f)$

Phase instability can also be normalized so that it is expressed as time instability by $S_x(f)$, the one-sided spectral density of the phase fluctuations expressed in units of time ($x(t)$):

$$S_x(f) = x_{rms}^2(f)\frac{1}{BW} \tag{A.13}$$

where, again, $x_{rms}(f)$ is defined in a fashion similar to that of $y_{rms}(f)$. From Equation (A.5), $S_\phi(f)$ and $S_x(f)$ are related by

$$S_x(f) = \frac{1}{(2\pi\nu_o)^2} S_\phi(f) \tag{A.14}$$

Since phase and frequency are directly related, that is, angular frequency is the time derivative of the phase, spectral densities of frequency and phase fluctuations are also related:

$$S_y(f) = (2\pi f)^2 S_x(f) = \frac{f^2}{{\nu_o}^2} S_\phi(f) \tag{A.15}$$

Other quantities related to phase instability are phase jitter and wander. The International Telecommunication Union (ITU) defines [timing] jitter and wander as variations [deviations] of the significant instants of a timing signal from their ideal positions in time excluding frequency offsets and drifts, where the jitter consists of variations with Fourier frequency above 10 Hz and the wander consists of variations with Fourier frequency below 10 Hz (see ITU-T Recommendation G.810 (08/96) [B56]). One can similarly define phase jitter through Equation (A.9). We define the phase jitter deviation by

$$\phi_{jitter} = \left[\int_{f_1}^{f_2} S_\phi(f)\mathrm{d}f \right]^{1/2} \tag{A.16}$$

which is the square root of the integral of $S_\phi(f)$ over the Fourier frequencies between a low frequency cutoff f_1 and high-frequency cutoff f_2 determined by the filtering properties of the application being considered. The use of 10 Hz for f_1 is useful for standardizing producers of time and frequency equipment, but care should be exercised because this cutoff is not necessarily related to user requirements. It is also noted that it is necessary to specify f_2 in order to properly define phase jitter deviation. Equation (A.16) assumes infinitely sharp filters. In practice this is hard to achieve and there can be contributions to the measured jitter deviation from phase noise outside the region of interest if the filter skirts are not steep enough.

Similarly, one can define phase wander deviation as the square root of the integral of $S_\phi(f)$ over Fourier frequencies below f_1, although it is noted that this quantity diverges when negative power law noise is present. It is also noted that is not possible to obtain $S_\phi(f)$ from the phase jitter deviation, unless the shape of $S_\phi(f)$ is known.

A.3 Time domain

In the time domain, an oscillator's frequency instability is defined by a two-sample deviation $\sigma_y(\tau)$, also called the *Allan deviation,* which is the square root of a two-sample variance $\sigma_y^2(\tau)$, also called the *Allan variance.* This variance $\sigma_y^2(\tau)$ assumes no dead time between adjacent average frequency samples. (*Dead time* refers to the time between time-ordered data sets when no measurement of frequency is taken.) For the averaging time τ,

$$\sigma_y(\tau) = \left[\frac{1}{2}\left\langle[\bar{y}_{k+1} - \bar{y}_k]^2\right\rangle\right]^{1/2} \tag{A.17}$$

where $t_k = t_0 + k\tau$ for some time origin t_0, $x_k = x(t_k)$, and

$$\bar{y}_k = \frac{1}{\tau}\int_{t_k}^{t_{k+1}} y(t)\mathrm{d}t = \frac{x(t_{k+1}) - x(t_k)}{\tau} = \frac{x_{k+1} - x_k}{\tau} \tag{A.18}$$

The symbol $\langle\ \rangle$ denotes an infinite time average, [i.e., in Equation (A.17) an average over $k = 1$ to $k = \infty$]. In practice, the requirement of infinite time average is never fulfilled, and the Allan deviation is estimated by

$$\sigma_y(\tau) \cong \left[\frac{1}{2(M-1)}\sum_{k=1}^{M-1}(\bar{y}_{k+1} - \bar{y}_k)^2\right]^{1/2} \tag{A.19}$$

where M is the number of frequency measurements. This is called the *non-overlapped estimate* of the Allan deviation because $y(t)$ is being averaged over non-overlapping intervals.

The estimate of the Allan deviation can also be expressed in terms of M+1 time deviation measurements $x_1,\ldots,x_{M+1}$ by combining Equation (A.18) and Equation (A.19):

$$\sigma_y(\tau) \cong \left[\frac{1}{2(M-1)\tau^2}\sum_{k=1}^{M-1}(x_{k+2} - 2x_{k+1} + x_k)^2\right]^{1/2} \tag{A.20}$$

Consistent, systematic effects such as frequency drift can be removed from the data before estimating the Allan deviation; see 3.1. Such data modifications shall be indicated.

If there is dead time between the frequency-deviation measurements and it is ignored in the computation of $\sigma_y(\tau)$, the resulting instability values will be biased (except for white frequency noise). Some of the biases have been studied and some correction tables published (see Barnes [B11], Lesage [B61], and Barnes and Allan [B13]). Therefore, the term $\sigma_y(\tau)$ shall not be used to describe such biased measurements without stating the bias together with $\sigma_y(\tau)$. The unbiased $\sigma_y(\tau)$ can be calculated from the biased values, using information in the references. Considering that $\{x_k\}$ can be routinely measured, it is preferred that $\{x_k\}$ is used to compute $\sigma_y(\tau)$ since the problem of dead time is avoided.

In general, estimates of $\sigma_y(\tau)$ with better confidence may be obtained using what is called *overlapped estimates*. Here it is assumed that the time deviation $x(t)$ is sampled with a fixed period τ_0. There are N samples $x_k = x(t_0 + k\tau_0)$, $k = 1, \ldots, N$. The estimate is obtained by computing

$$\sigma_y(\tau) \cong \left[\frac{1}{2(N-2m)\tau^2}\sum_{k=1}^{N-2m}(x_{k+2m} - 2x_{k+m} + x_k)^2\right]^{1/2} \tag{A.21}$$

where m is a positive integer and $\tau = m\tau_0$.

Examples of overlapped $\sigma_y(\tau)$ estimates are given in Annex C.

Equation (A.21) shows that $\sigma_y(\tau)$ acts like a second-difference operator on the time deviations usually providing a stationary measure of the stochastic behavior even for nonstationary processes. An efficient spacing of τ values in a plot of log $\sigma_y(\tau)$ vs. log τ sets $m = 2^p$, where $p = 0, 1, 2, 3, \ldots$.

When differentiating between white and flicker PM noise is desirable, a *modified deviation*, denoted as *Mod* $\sigma_y(\tau)$, may be used to characterize frequency instabilities (see Allan and Barnes [B4], Stein [B89]). Unlike $\sigma_y(\tau)$, *Mod* $\sigma_y(\tau)$ has the property of yielding different dependence on τ for white phase noise and flicker PM; the dependencies are $\tau^{-3/2}$ and τ^{-1}, respectively. (The dependence for $\sigma_y(\tau)$ is approximately τ^{-1} for both white and flicker PM.) Another advantage is that *Mod* $\sigma_y(\tau)$ averages wideband PM faster than τ^{-1}. *Mod* $\sigma_y(\tau)$ is defined for τ_0 -sampled time deviations $x_k = x(t_0 + k\tau_0)$, as

$$Mod\ \sigma_y(\tau) = \left\{ \frac{1}{2\tau^2} \left\langle \left[\frac{1}{m} \sum_{i=1}^{m} (x_{i+2m} - 2x_{i+m} + x_i) \right]^2 \right\rangle \right\}^{1/2} \tag{A.22}$$

where $\tau = m\tau_0$ for an integer m, and $\langle\ \rangle$ indicates an infinite time average over t_0. In practice, the modified Allan deviation is estimated from N samples $x_1, \ldots, x_N$ by

$$Mod\ \sigma_y(\tau) \cong \left\{ \frac{1}{2\tau^2 m^2 (N-3m+1)} \sum_{j=1}^{N-3m+1} \left[\sum_{i=j}^{m+j-1} (x_{i+2m} - 2x_{i+m} + x_i) \right]^2 \right\}^{1/2} \tag{A.23}$$

For examples of $\sigma_y(\tau)$ and *Mod* $\sigma_y(\tau)$ see Annex C.

A measure of rms time deviation that is often used in time transfer systems, such as the global positioning system (GPS), is $\sigma_y(\tau)$, defined as

$$\sigma_x(\tau) = \frac{\tau}{\sqrt{3}}\, Mod\ \sigma_y(\tau) \tag{A.24}$$

This quantity is useful when white and flicker PM noise dominate a synchronization system.

Another approach to distinguish different noise types is to use multivariance analysis (see Vernotte et al. [B93]). By using several variances to analyze the same data it is possible to estimate the coefficients for the five noise types. For a description of noise types see Annex B.

At long averaging times, greater than 10% of the total measurement time (i.e., $\tau > 0.1N\tau_0$), an Allan deviation estimate has potential errors and a bias related to its insensitivity to odd (antisymmetric) noise processes in *x(t)* [odd about the midpoint of the *x(t)* data or even in terms of average $\overline{y}_k$]. This insensitivity to odd noise processes is illustrated in Figure A.2. Part a) of Figure A.2 shows three phase samples of a noise process that is odd about x_2. The calculated fractional frequency deviations according to Equation

(A.19) are shown in part b) of Figure A.2. Since $\overline{y}_1$ and $\overline{y}_2$ are equal, contributions due to this noise process will not show up in $\sigma_y(\tau)$.

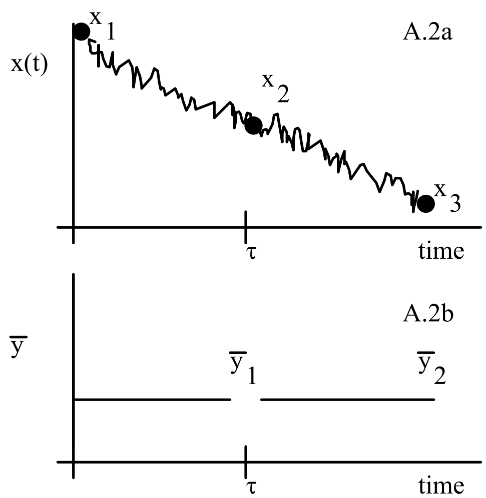

Figure A.2—Odd noise process about x_2

For this reason, when τ exceeds 10% of the data sample, using the total deviation $\hat{\sigma}_{y,TOTAL}(\tau)$ to estimate $\sigma_y(\tau)$ is recommended. $\hat{\sigma}_{y,TOTAL}(\tau)$ extends the x_k sequence at both ends by reflection about the endpoints to provide a better estimate of frequency stability. The advantages of $\hat{\sigma}_{y,TOTAL}(\tau)$ are outlined in several references (see Howe [B41], [B42], and Howe and Greenhall [B44]).

To define $\hat{\sigma}_{y,TOTAL}(\tau)$ let $x_1, ..., x_N$ $(N \geq 5)$ be the time-residual data, sampled with time period τ_0 and let the maximum value of τ be $m_{\max}\tau_0$, where $m_{\max}$ is the integer part of $(N - 1)/2$. We construct a new, longer sequence $\{ x'_k : 2 - m_{\max} \leq k \leq N + m_{\max} - 1 \}$ as follows:

$$x'_1 = x_1, \quad x'_2 = x_2, \quad \ldots, \quad x'_N = x_N$$

$$x'_0 = 2x_1 - x_2, \quad x'_{-1} = 2x_1 - x_3, \quad \ldots, \quad x'_{2-m_{\max}} = 2x_1 - x_{m_{\max}}$$

$$x'_{N+1} = 2x_N - x_{N-1}, \quad x'_{N+2} = 2x_N - x_{N-2}, \quad \ldots, \quad x'_{N+m_{\max}-1} = 2x_N - x_{N-m_{\max}+1}$$

Equation (A.21) with $\tau = m\tau_0$ $(m \leq m_{\max})$ can be applied to the sequence $x'_{2-m}, \ldots, x'_{N+m-1}$ to define

$$\hat{\sigma}_{y,TOTAL}(\tau) = \left(\frac{1}{2\tau^2(N-2)} \sum_{k=2}^{N-1} [x'_{k-m} - 2x'_k + x'_{k+m}]^2 \right)^{1/2} \tag{A.25}$$

$\hat{\sigma}_{y,TOTAL}(\tau)$ can also be represented in terms of extended normalized frequency fluctuation averages as

$$\hat{\sigma}_{y,TOTAL}(\tau) = \left[\frac{1}{2(N-2)} \sum_{k=2}^{N-1} (\bar{y}'_k - \bar{y}'_{k-m})^2 \right]^{\frac{1}{2}} \tag{A.26}$$

where $\bar{y}'_k = (x'_{k+m} - x'_k)/\tau$.

A.4 Systematic instabilities

The long-term frequency change of a source is called *frequency drift.* Drift includes frequency changes caused by changes in the components of the oscillator, in addition to sensitivities to the oscillator's changing environment and changes caused by load and power supply changes (see Vig and Meeker [B96]).

The frequency aging of an oscillator refers to the change in the frequency of oscillation caused by changes in the components of the oscillator, either in the resonant unit or in the accompanying electronics. Aging differs from drift in that it does not include frequency changes due to changes in the environment, such as temperature changes. Aging is thus a measure of the long-term stability of the oscillator, independent of its environment. The frequency aging of a source (positive or negative) is typically *maximum* immediately after turn-on.

Aging can be specified by the normalized rate of change in frequency at a specified time after turn-on (for example, 1×10^{-10} per day after 30 days), or by the total normalized change in frequency in a period of time (for example, 1×10^{-8} per month) (see Vig and Meeker [B96]). It is worth noting that some clocks can have a very long (multi-year) equilibration following turn-on before frequency aging takes on an essentially unchanging linear rate (see Camparo, Klimcak, Herbulock [B25]).

A.5 Clock-time prediction

The time difference between a real clock and an ideal uniform time scale, also known as time interval error (*TIE*), observed over a time interval starting at time t_0 and ending at $t_0 + t$ is defined as

$$TIE(t) = x(t_0 + t) - x(t_0) = \int_{t_0}^{t_0+t} y(t')\mathrm{d}t' \tag{A.27}$$

For fairly simple models, regression analysis can provide efficient estimates of the *TIE* (see Draper and Smith [B32] and CCIR [B26]). In general, there are many estimators possible for any statistical quantity. An efficient and unbiased estimator is preferred. Using the time domain measure $\sigma_y(\tau)$, the following estimate of the standard deviation (rms) of *TIE* and its associated systematic departure due to a linear frequency drift (plus its uncertainty) can be used to predict a probable *TIE* of a clock synchronized at time t_0 and left free running thereafter:

$$\mathrm{RMS}\, TIE_{\mathrm{est}}(t) = t\left[\left(\frac{\sigma_{x_0}}{t}\right)^2 + \sigma_{y_0}^2 + \sigma_y^2(t) + \frac{a^2}{4}t^2\right]^{1/2} \tag{A.28}$$

where

σ_{x_0} is the uncertainty in the initial synchronization

σ_{y_0}	is the uncertainty in the initial frequency adjustment
$\sigma_y(t)$	is the two-sample deviation describing the random frequency instability of the clock at $\tau = t$ computed after a linear frequency drift has been remove
a	is the normalized linear frequency drift per unit of time plus the uncertainty in the drift estimate

The third term in the brackets provides an optimum and unbiased estimate [under the condition of an optimum (rms) prediction method] in the cases of white noise frequency modulation (FM) and/or random walk FM. The third term is too optimistic, by about a factor of 1.4, for flicker FM noise, and too pessimistic, by about a factor of 3, for white PM noise (see Barnes and Allan [B12] and Allan [B3]).

This estimate is a useful and fairly simple approximation. A more complete error analysis becomes difficult. If carried out, such an analysis needs to include the methods of time prediction, the uncertainties of the clock parameters using the confidence limits of measurements defined as follows, the detailed clock noise models, systematic effects, etc.

A quantity often used to characterize the stability of clocks in telecommunication systems is the maximum time interval error (*MTIE*). *MTIE* is defined as the maximum time difference minus the minimum time difference between a clock and an ideal reference (see Bregni [B23]).

Annex B

(informative)

Power-laws and conversion between frequency and time domain

B.1 Power-law spectral densities

Power-law spectral densities serve as reasonable and accurate models of the random fluctuations in precision oscillators. In practice, these random fluctuations can often be represented by the sum of five such noise processes assumed to be independent, as

$$S_y(f) = \begin{cases} \sum_{\alpha=-2}^{+2} \mathrm{h}_\alpha f^\alpha & \text{for } 0 < f < f_h \\ 0 & \text{for } f \geq f_h \end{cases} \tag{B.1}$$

where

h_α is constant
α is integer
f_h is high-frequency cutoff of an infinitely sharp low-pass filter

High-frequency divergence is eliminated by the restrictions on f in this equation. The identification and characterization of the five noise processes are given in Table B.1, and shown in Figure B.1.

Table B.1—Functional characteristics of the independent noise processes used in modeling frequency instability of oscillators

Description of noise process	Slope characteristics of log-log plot				
	Frequency domain		Time domain		
	$S_y(f)$	$S_\phi(f)$ or $S_x(f)$	$\sigma_y^2(\tau)$	$\sigma_y(\tau)$	$Mod\ \sigma_y(\tau)$
	α	β	μ	$\mu/2$	μ'
Random walk FM	–2	–4	1	1/2	1/2
Flicker FM	–1	–3	0	0	0
White FM	0	–2	–1	–1/2	–1/2
Flicker PM	1	–1	–2	–1	–1
White PM	2	0	–2	–1	–3/2

$$S_y(f) = \frac{(2\pi f)^2}{(2\pi \nu_0)^2} S_\phi(f) = \mathrm{h}_\alpha f^\alpha$$

$\sigma_y^2(\tau) \sim |\tau|^\mu$

$$S_\phi(f) = \nu_0^2 \mathrm{h}_\alpha f^{\alpha-2} = \nu_0^2 \mathrm{h}_\alpha f^\beta \quad (\beta \equiv \alpha - 2)$$

$\sigma_y(\tau) \sim |\tau|^{\mu/2}$

$$S_x(f) = \frac{1}{4\pi^2} \mathrm{h}_\alpha f^{\alpha-2} = \frac{1}{4\pi^2} \mathrm{h}_\alpha f^\beta$$

$Mod\ \sigma_y(\tau) \sim |\tau|^{\mu'}$

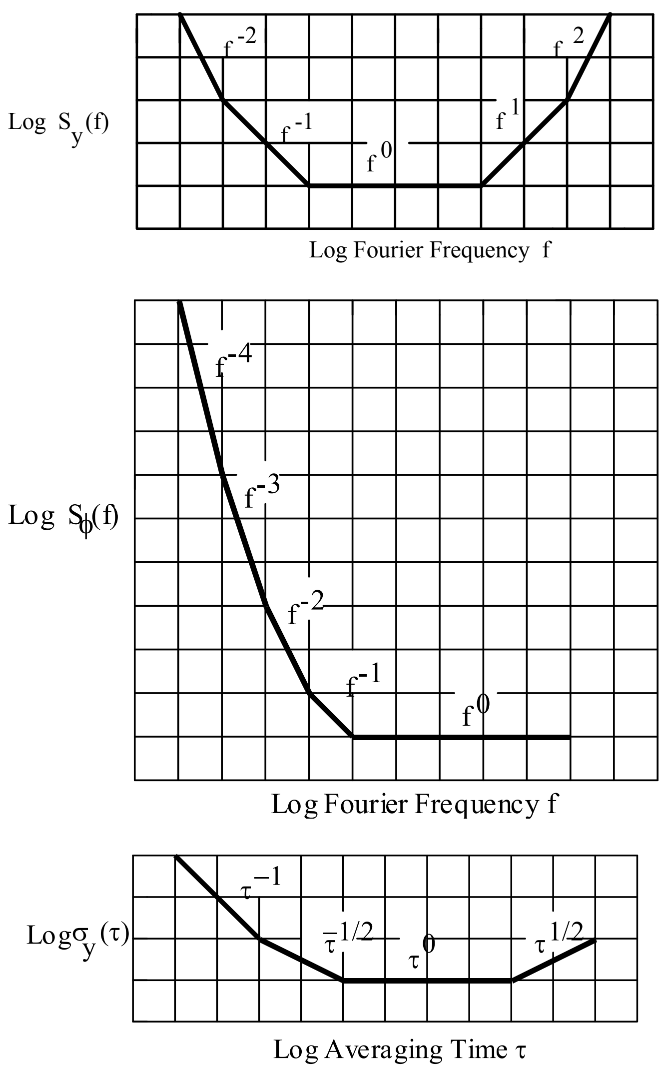

Figure B.1—Slope characteristics of the five independent noise processes

B.2 Conversion between frequency and time domain

The operation of the counter, averaging the frequency for a time τ, may be thought of as a filtering operation. The transfer function $H(f)$ of this equivalent filter is then the Fourier transform of the impulse response function of the filter. The time domain frequency instability is then given by

$$\sigma_y(M,T,\tau) = \left[\int_0^{\infty} S_y(f) \left| H(f) \right|^2 \mathrm{d}f \right]^{1/2} \tag{B.2}$$

where $S_y(f)$ is the one-sided spectral density of normalized frequency fluctuations. $1/T$ is the measurement rate ($T-\tau$ is the dead time between measurements). In the case of the two-sample deviation $|H(f)|^2 = 2(\sin^4 \pi\tau f)/(\pi\tau f)^2$. The two-sample deviation can thus be computed from

$$\sigma_y(\tau) = \left[2 \int_0^{f_h} S_y(f) \frac{\sin^4(\pi\tau f)}{(\pi\tau f)^2} \mathrm{d}f \right]^{1/2} \tag{B.3}$$

Specifically, for the compound power law model given in Equation (B.1), the time domain measure is given by

$$\sigma_y(\tau) = \left[\mathrm{h}_{-2} \frac{(2\pi)^2}{6} \tau + \mathrm{h}_{-1} 2 \ln 2 + \mathrm{h}_0 \frac{1}{2\tau} + \mathrm{h}_1 \frac{1.038 + 3 \ln(2\pi f_h \tau)}{(2\pi)^2 \tau^2} + \mathrm{h}_2 \frac{3 f_h}{(2\pi)^2 \tau^2} \right]^{1/2} \tag{B.4}$$

Equation (B.4) assumes that f_h is the high-frequency cutoff of an infinitely sharp low-pass filter and that $2\pi f_h \tau >> 1$. This equation also implicitly assumes that the random driving mechanism for each term is independent of the others, and that the mechanism is valid over all Fourier frequencies. These assumptions may not always be true.

The modified two-sample deviation can also be computed from $S_y(f)$ by using

$$Mod\sigma_y(\tau) = \left[\frac{2}{m^4} \int_0^{f_h} S_y(f) \frac{\sin^6(\pi\tau f)}{(\pi\tau_0 f)^2 \sin^2(\pi\tau_0 f)} \mathrm{d}f \right]^{1/2} \tag{B.5}$$

where m is a positive integer and $\tau = m\tau_0$ (see Bernier [B18]).

Table B.2 gives the coefficients of the translation from $S_y(f)$ (frequency domain) to $\sigma_y^2(\tau)$ (time domain). In general computation of $S_y(f)$ or related frequency domain measurements from $\sigma_y(\tau)$ or $Mod\ \sigma_y(\tau)$ are not permitted unless only one power law noise type is present. Nevertheless, when several noise types are present, special analysis can be made on the time-domain data to obtain the coefficients (in the frequency domain) for each power law (see Vernotte et al. [B93]).

Table B.2—Translation of frequency instability measures from spectral densities in frequency domain to variances in time domain for an infinitely sharp low-pass filter with $2\pi f_h \tau >> 1$

Description of noise process	$S_y(f) =$	$S_\phi(f) =$	$\sigma_y^2(\tau) =$
Random walk FM	$h_{-2} f^{-2}$	$h_{-2} \nu^2 f^{-4}$	$Ah_{-2} \tau^1$
Flicker FM	$h_{-1} f^{-1}$	$h_{-1} \nu^2 f^{-3}$	$Bh_{-1} \tau^0$
White FM	$h_0 f^0$	$h_0 \nu^2 f^{-2}$	$Ch_0 \tau^{-1}$
Flicker PM	$h_1 f^1$	$h_1 \nu^2 f^{-1}$	$Dh_1 \tau^{-2}$
White PM	$h_2 f^2$	$h_2 \nu^2 f^0$	$Eh_2 \tau^{-2}$

$$A = \frac{2\pi^2}{3} \qquad B = 2\ln 2 \qquad C = 1/2$$

$$D = \frac{1.038 + 3\ln(2\pi f_h \tau)}{4\pi^2} \qquad E = \frac{3 f_h}{4\pi^2}$$

Annex C

(informative)

Examples of computation of deviations

C.1 Introduction

This annex contains basic examples on how to compute the deviations used to describe frequency instabilities in the time domain. For more information on this topic and on how to assess the validity of the computations when using larger number of samples, see Riley [B78] and [B77].

C.2 Allan deviation $\sigma_y(\tau)$ examples

Figure C.1 shows a plot of the time deviation between a pair of oscillators as a function of time. The recorded time samples for $\tau = 1$ s are shown in the first column of Table C.1. To compute $\sigma_y(\tau = 1\text{ s})$, compute the average fractional frequency deviation for x_ks separated by 1 s, then calculate the difference between adjacent $\bar{y}_k$s and use Equation (A.19) to obtain $\sigma_y(\tau)$. See Table C.1.

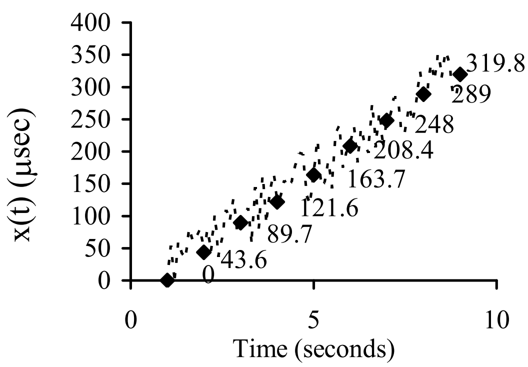

Figure C.1—Plot of $x(t)$ between a pair of oscillators

Table C.1—Steps to compute $\sigma_y(1\text{ s})$

k	x_k μs	$\bar{y}_k = (x_{k+1}-x_k)/\tau$ $\times 10^{-6}$	$\bar{y}_{k+1}-\bar{y}_k$ $\times 10^{-6}$
1	0	43.6	2.5
2	43.6	46.1	–14.2
3	89.7	31.9	10.2
4	121.6	42.1	2.6
5	163.7	44.7	–5.1
6	208.4	39.6	1.4
7	248	41.0	–10.2
8	289	30.8	—
9	319.8	—	—

In this example $N = 9$ (number of time samples) and $M = 8$; therefore

$$\sigma_y(\tau = 1\mathrm{s}) = \left[\frac{1}{2(7)}\sum_{k=1}^{7}\left(\bar{y}_{k+1} - \bar{y}_k\right)^2\right]^{1/2} = 5.67\times 10^{-6} \tag{C.1}$$

For $\tau = 2$ s ($= 2\tau_0$) the procedure is similar: compute the fractional frequency deviation for x_ks separated by 2 s, then calculate the difference between adjacent $\bar{y}_k$s and use Equation (A.19) to obtain $\sigma_y(\tau)$. See Table C.2.

Table C.2—Steps for computing σ_y(2 s)

k	x_k μs	$\bar{y}_k = (x_{k+2} - x_k)/\tau$ $\times 10^{-6}$	$\bar{y}_{k+2} - \bar{y}_k$ $\times 10^{-6}$
1	0	44.85	–7.85
2	43.6	—	—
3	89.7	37	5.15
4	121.6	—	—
5	163.7	42.15	–6.25
6	208.4	—	—
7	248	35.9	—
8	289	—	—
9	319.8	—	—

For this example $M = 4$ since there are a total of four $\bar{y}_k$s. Therefore

$$\sigma_y(\tau = 2\mathrm{s}) = \left[\frac{1}{2(3)}\sum_{k=1}^{3}\left(\bar{y}_{k+m} - \bar{y}_k\right)^2\right]^{1/2} = 4.6\times 10^{-6} \tag{C.2}$$

As mentioned in A.3, it is usually more efficient to use overlapped estimates when possible since this results in a better confidence interval. Figure C.2 illustrates how to compute the $\bar{y}_k$s for an overlapped estimate of $\sigma_y(\tau = 2$ s). In this case $m = 2$ ($\tau = 2\tau_0$) and $M = 8$. The $\bar{y}_k$s and the second difference values are shown in Table C.3.

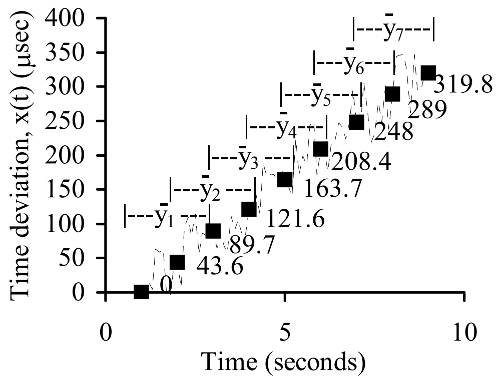

Figure C.2—Computation of $\bar{y}_k$s for overlapped estimates

Table C.3—Steps for computing an overlapped estimate of σ_y(2 s)

k	x_k µs	$\bar{y}_k = (x_{k+2} - x_k)/\tau$ $\times 10^{-6}$	$\bar{y}_{k+2} - \bar{y}_k$ $\times 10^{-6}$
1	0	44.85	–7.85
2	43.6	39	4.4
3	89.7	37	5.15
4	121.6	43.4	–3.1
5	163.7	42.15	–6.25
6	208.4	40.3	—
7	248	35.9	—
8	289	—	—
9	319.8	—	—

There are a total of (N – 2m = 5) second difference values ($\bar{y}_{k+m} - \bar{y}_k$); therefore the Allan deviation equation becomes

$$\sigma_y(\tau) = \left[\frac{1}{2(N-2m)} \sum_{k=1}^{N-2m} (\bar{y}_{k+m} - \bar{y}_k)^2 \right]^{1/2} \tag{C.3}$$

where $\bar{y}_k = (x_{k+m} - x_k)/\tau$. Equation (C.3) becomes Equation (A.21) when the $\bar{y}_k$s are expressed in terms of the initial time residual measurements. It is used in this example to help explain the origin of Equation (A.21). Using the values in Table C.3 (last column), Equation (C.3) yields

$$\sigma_y(2\text{s}) = \left\{ \frac{1}{2(5)} [(-7.85)^2 + 4.4^2 + 5.15^2 + (-3.1)^2 + (-6.25)^2] \right\}^{1/2} = 3.95 \times 10^{-6}$$

C.3 Modified Allan deviation *Mod* $\sigma_y(\tau)$ example

The *modified Allan deviation* may also be used to characterize frequency stability in the time domain when differentiating between white and flicker PM noise is desirable. This deviation uses the average of m adjacent x_ks when computing the stability for $\tau = m\tau_0$. The fractional frequency deviations are then obtained using the $\bar{x}_k$s. See Figure C.3 for computation of $\bar{x}_k$s and $\bar{y}'_k$s.

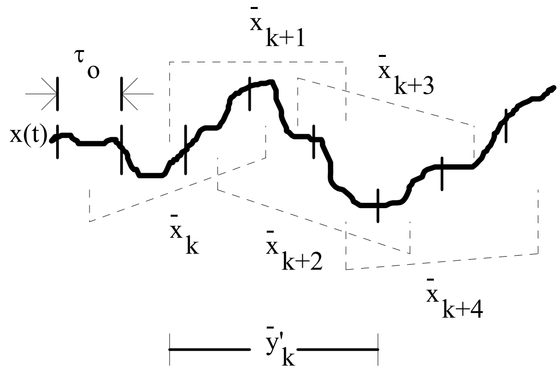

Figure C.3—Method for calculating $\bar{x}_k$s, and $\bar{y}'_k$s for *Mod* $\sigma_y(\tau)$

Table C.4 shows the computed $\bar{x}_k$s, and $\bar{y}'_k$s for τ = 2s. The modified Allan deviation can then be obtained by using Equation (A.19) and the fact that m = 2 and the equivalent M is $N - 3m + 1$:

$$Mod\ \sigma_y(\tau) = \left\{ \frac{1}{2(N-3m+1)} \sum_{k=1}^{N-3m+1} (\bar{y}'_{k+m} - \bar{y}'_k)^2 \right\}^{1/2} \tag{C.4}$$

$$Mod\ \sigma_y(2s) = \left\{ \frac{1}{2(4)} [(-1.73)^2 + 4.78^2 + 1.03^2 + (-4.68)^2] \right\}^{1/2} = 2.47 \times 10^{-6}$$

Equation (C.4) becomes Equation (A.23) when expressing the $\bar{y}'_k$s in terms of the initial time residual measurements. It is used in this example to help explain the origin of Equation (A.23).

Table C.4—Computed $\bar{x}_k$, and $\bar{y}'_k$ values for *Mod* σ_y(2 s)

k	x_k μs	$\bar{x}_k = (x_{k+1} + x_k)/2$ μs	$\bar{y}'_k = (\bar{x}_{k+2} - \bar{x}_k)/\tau$ $\times 10^{-6}$	$\bar{y}'_{k+2} - \bar{y}'_k$ $\times 10^{-6}$
1	0	21.8	41.93	–1.73
2	43.6	66.65	38	4.78
3	89.7	105.65	40.2	1.03
4	121.6	142.65	42.78	–4.68
5	163.7	186.05	41.23	—
6	208.4	228.2	38.1	—
7	248	268.5	—	—
8	289	304.4	—	—
9	319.8	—	—	—

C.4 Total deviation $\hat{\sigma}_{y,TOTAL}(\tau)$ example

In Annex A it was recommended that the total deviation $\hat{\sigma}_{y,TOTAL}(\tau)$ be used to characterize fractional frequency fluctuations when τ exceeds 10% of the data sample. $\hat{\sigma}_{y,TOTAL}(\tau)$ extends the x_k sequence at both ends by reflection about the endpoints to provide a better estimate of frequency stability.

As an example we will compute $\hat{\sigma}_{y,TOTAL}(2\text{s})$ for $x(t)$ in Figure C.4. This data set is different from the one used in the previous examples. In this case x_1, x_3, and x_5 almost fall into a line; therefore the value for $\sigma_y(\tau)$ will be negatively biased (too optimistic).

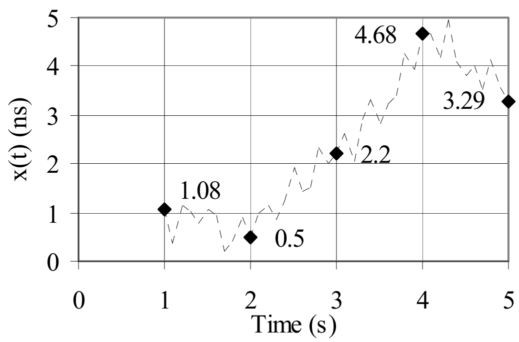

Figure C.4—*x*(*t*) as a function of time

For this data set, $N = 5$ and $m_{\max} = (5 - 1)/2 = 2$. There are only two extra x'_k to adjoin, namely, $x'_0 = 2x_1 - x_2 = 1.66$ ns, $x'_6 = 2x_5 - x_4 = 1.90$ ns. According to Equation (A.26)

$$\hat{\sigma}_{y,TOTAL}(2\text{s}) = \left\{ \frac{1}{2(3)} \left[(\bar{y}'_2 - \bar{y}'_0)^2 + (\bar{y}'_3 - \bar{y}'_1)^2 + (\bar{y}'_4 - \bar{y}'_2)^2 \right] \right\}^{1/2}$$

where

$$\bar{y}'_0 = \frac{(x'_2 - x'_0)}{2} = \frac{(0.50 - 1.66)}{2} = -0.58 \times 10^{-9}$$

$$\bar{y}'_1 = \frac{(x'_3 - x'_1)}{2} = \frac{(2.20 - 1.08)}{2} = 0.56 \times 10^{-9}$$

$$\bar{y}'_2 = \frac{(x'_4 - x'_2)}{2} = \frac{(4.68 - 0.50)}{2} = 2.09 \times 10^{-9}$$

$$\bar{y}'_3 = \frac{(x'_5 - x'_3)}{2} = \frac{(3.29 - 2.20)}{2} = 0.545 \times 10^{-9}$$

$$\bar{y}'_4 = \frac{(x'_6 - x'_4)}{2} = \frac{(1.90 - 4.68)}{2} = -1.39 \times 10^{-9}$$

Therefore

$$\hat{\sigma}_{y,TOTAL}(2\mathrm{s}) = \left\{ \frac{1}{6}\left[(2.67)^2 + (0.015)^2 + (3.48)^2 \right] \right\}^{1/2} \times 10^{-9} = 1.79 \times 10^{-9}$$

This value can be compared to the value obtained for the Allan deviation:

$$\sigma_y(2\mathrm{s}) = \left(\frac{1}{2}[\bar{y}_3 - \bar{y}_1]^2 \right)^{1/2} = \frac{1}{\sqrt{2}}\left|0.545 - 0.56\right| \times 10^{-9} = 1.06 \times 10^{-11}$$

(Note that $\bar{y}_1 = \bar{y}'_1$, $\bar{y}_3 = \bar{y}'_3$.) σ_y(2 s) is seriously negatively biased by two orders of magnitude compared to $\hat{\sigma}_{y,TOTAL}(2\mathrm{s})$.

A slight negative bias in $\hat{\sigma}_{y,TOTAL}(\tau)$ has been found for flicker FM noise and random walk FM noise. It is possible to remove this bias if the noise type is assumed to be known (see Howe and Greenhall [B44]).

Annex D

(informative)

Other variances deviations that have been used to describe frequency instabilities in the time domain

A variety of deviations and error measures other than $\sigma_y(\tau)$, Mod $\sigma_y(\tau)$, $\sigma_x(\tau)$, and rms $TIE_{\text{est}}(t)$ have been used in this field and are being used by other fields, societies, and organizations. Other deviations of y that have been introduced are ones based on a structure function approach (see Lindsey and Chie [B67]), a high-pass deviation (see Rutman [B83]), and two versions of the *Hadamard* deviation. An earlier three-sample Hadamard deviation, which is based on Hadamard's original work, has been defined as (see Rutman [B83] and Baugh [B16]):

$$\sigma_H(N=3, T=\tau, \tau) = \left[< [\bar{y}_3 - \bar{y}_2 + \bar{y}_1]^2 > \right]^{1/2} \tag{D.1}$$

where $\bar{y}_k$ is given by Equation (A.18). A later, different version, which is a 2nd order difference variance of $y(t)$ (see Reinhardt [B76]), is defined as (see Hutsell [B48] and Riley [B81]):

$$_H\sigma_y(\tau) = \left[(1/6) < [\bar{y}_3 - 2\bar{y}_2 + \bar{y}_1]^2 > \right]^{1/2} \tag{D.2}$$

This later version has been introduced in "total" form as a useful predictor of GPS clock error (see Howe et al. [B46]). A "modified" version of the Hadamard deviation is described in Bregni and Jmoda [B24]. More information on the Hadamard deviation can be found in Wan, Visr, Roberts [B103], Walter [B101], Boileau and Picinbono [B21], Howe et al. [B45], Greenhall and Riley [B38], Gagnepain [B35], and Riley [B81].

Other deviations of x and *TIE* are also in use in other fields and by other societies and organizations. Such deviations are the standard deviation of x, its N-sample statistic, the *TIE* deviation $TIErms\,(\tau)$, and its N-sample statistic (see IEEE Std 1057™-1994 [B53], IEEE Std 181™-2003 [B52], IEEE Std 1241™-2000 [B55], ITU-T Recommendation G.810 (8/96) [B56]). The definitions and properties of these deviations are listed in Table D.1. Such deviations can be written in a spectral form similar to Equation (B.3) as

$$\sigma = \left[\int_0^{f_h} K_y(f) S_y(f) \mathrm{d}f \right]^{1/2} \tag{D.3}$$

where $K_y(f)$ is a y-kernel that defines the spectral properties of the deviation under consideration relative to $S_y(f)$. The y-kernels of the above deviations are listed in Table D.1.

The deviation integral in Equation (D.3) can diverge at $f = 0$ for a given $K_y(f)$ and $S_y(f)$; this implies that the chosen stability deviation statistic does not converge to a stable value for this kind of oscillator noise. One can apply additional high-pass filtering to the noise before measuring its stability deviation, or choose a different stability deviation that converges for the given noise. More information on these deviations can be found in Reinhardt [B74], [B75], [B76].

Table D.1—Deviations of *x*(*t*) and *TIE*

Deviation σ name	**σ *t*-domain definition ($x_k = x(t+k\tau_0)$, $\tau = m\tau_0$)**	**$K_y(f)$** $\sigma = \left[\int_0^{f_h} K_y(f) S_y(f)\mathrm{d}f\right]^{1/2}$	**α range for convergence (using f_h)** $S_y(f) = \mathrm{h}_\alpha f^\alpha$
Standard deviation of x $\sigma_{x\text{-std}}$	$\left[\left\langle [x(t)]^2 \right\rangle\right]^{1/2}$	$(2\pi f)^{-2}$ small f: $\propto f^{-2}$ (Reinhardt [B76])	$\alpha \geq 2$
Sample deviation of x $\sigma_{x\text{-}N\text{-sample}}$	$\left[(N-1)^{-1}\sum_{k=1}^{N}[x_k - M_k]^2\right]^{1/2}$ $M_x = N^{-1}\sum_{k=1}^{N} x_k$	$(2\pi f)^{-2}(1 - \mathrm{sinc}^2(\pi f N\tau))$ $[f\tau << 1, N >> 1]$ small f: $\propto 1$ (Reinhardt [B76])	$\alpha \geq 0$
TIErms(τ) (ITU-T G.810 [B56])	$\left[\left\langle [x(t+\tau) - x(t)]^2 \right\rangle\right]^{1/2}$	$\tau^2\mathrm{sinc}^2(\pi f\tau)$ small f: $\propto 1$	$\alpha \geq 0$
TIErms(τ) *N*-sample statistic (ITU-T G.810 [B56])	$\left[(N-m)^{-1}\sum_{k=1}^{N-m}[x_{k+m} - x_m]^2\right]^{1/2}$	$\tau^2\mathrm{sinc}^2(\pi f\tau)$ small f: $\propto 1$	$\alpha \geq 0$

Annex E

(informative)

Confidence limits of measurements

A simple method to compute the confidence interval for $\sigma_y(\tau)$ (see Lesage and Audoin [B63], [B65]), which assumes a symmetric (Gaussian) distribution, uses the relation

$$I_\alpha = \sigma_y(\tau)\kappa_\alpha M^{-1/2} \tag{E.1}$$

where

I_α	is the uncertainty of the estimate
κ_α	is a constant
α	is an integer that depends on the type of noise (see Annex B)
M	is the number of non-overlapped τ-averaged frequency samples used in the estimate

The confidence limits are $\sigma_y(\tau) \pm I_\alpha$.

For a 1σ or 68 % confidence interval the values for κ_α are as follows:

$\kappa_2 = 0.99$

$\kappa_1 = 0.99$

$\kappa_0 = 0.87$

$\kappa_{-1} = 0.77$

$\kappa_{-2} = 0.75$

As an example of the Gaussian model with $M = 100$, $\alpha = -1$ (flicker frequency noise) and $\sigma_y(\tau = 1\text{ s}) = 1 \times 10^{-12}$, one may write

$I_\alpha \cong \sigma_y(\tau) \times (0.77) \times (100)^{-1/2} = \sigma_y(\tau) \times (0.077)$

which gives

$\sigma_y(\tau = 1\text{ s}) = (1 \pm 0.08)\ 10^{-12}$

This analysis for $\sigma_y(\tau)$ applies only to the non-overlapped estimate, Equation (A.19), and is valid only for $M \geq 10$. If M is small, then the plus and minus confidence limits become sufficiently asymmetric and the κ_α coefficients are not valid. However, these confidence limits can be calculated (see Lesage and Audoin [B63]).

Another way of computing confidence intervals for $\sigma_y(\tau)$ is to use the chi-squared distribution. The estimated Allan variance has a chi-squared distribution function given by Equation (E.2). The number of degrees of freedom for a specific noise process and number of samples can be computed and then used in Equation (E.2) to compute the confidence interval (see Howe, Allan, Barnes [B43]):

$$\chi^2 = (df)\frac{\hat{\sigma}_y^2}{\sigma_y^2} \tag{E.2}$$

where

df	is the number of degrees of freedom
$\hat{\sigma}_y^2$	is the estimated (measured) Allan variance
σ_y^2	is the true Allan variance

Table E.1 shows empirical equations to compute the number of degrees of freedom for different types of noise processes (see Howe, Allan, Barnes [B43]). This table is valid only for overlapped estimates of the Allan variance, Equation (A.21) squared.

To compute the confidence interval for $\hat{\sigma}_y$ ($\tau = 1$ s) = 10^{-12}, for flicker frequency noise, $N = 101$, and τ_o = 0.5 s ($m = 2$), we first find the number of degrees of freedom using Table E.1:

$$df = \frac{5N^2}{4m(N+3m)} \tag{E.3}$$

$$= \frac{5(101^2)}{4(2)(101+3(2))} = 59.6$$

For a 1σ (68%) confidence interval, the χ^2 values needed are $\chi^2(0.16)$ and $\chi^2(1\text{-}0.16)$. These values can be obtained from numerical tables of the chi-squared distribution function or from various computer programs. For 59 degrees of freedom, $\chi^2(0.16) = 48.25$ and $\chi^2(0.84) = 69.73$.

Therefore

$$\chi^2(0.16) < \frac{59.6\hat{\sigma}_y^2}{\sigma_y^2} < \chi^2(0.84) \tag{E.4}$$

or

$$\frac{59.6\hat{\sigma}_y^2}{\chi^2(0.84)} < \sigma_y^2 < \frac{59.6\hat{\sigma}_y^2}{\chi^2(0.16)} \tag{E.5}$$

$$0.92\hat{\sigma}_y < \sigma_y < 1.11\hat{\sigma}_y \tag{E.6}$$

Table E.1—Empirical equations for the number of degrees of freedom of the Allan variance estimate [a]

Noise process	Degrees of freedom [b]
White PM	$\dfrac{(N+1)(N-2m)}{2(N-m)}$
Flicker PM	$\exp\left[\ln\left(\dfrac{N-1}{2m}\right)\ln\left(\dfrac{(2m+1)(N-1)}{4}\right)\right]^{1/2}$
White FM	$\left[\dfrac{3(N-1)}{2m}-\dfrac{2(N-2)}{N}\right]\dfrac{4m^2}{4m^2+5}$
Flicker FM	$\dfrac{2(N-2)^2}{2.3N-4.9}$ for $m=1$ $\dfrac{5N^2}{4m(N+3m)}$ for $m \geq 2$
Random-walk FM	$\dfrac{N-2}{m}\dfrac{(N-1)^2-3m(N-1)+4m^2}{(N-3)^2}$

[a] Adapted from Howe, Allan, Barnes [B43].
[b] N = number of samples, and $m = \tau/\tau_0$

Other methods have been developed for computing the confidence interval of *Mod* $\sigma_y(\tau)$ (see Walter [B102] and Greenhall [B36]. Table E.2 shows a comparison of confidence intervals for $\sigma_y(\tau)$ (no overlap and full overlap) and *Mod* $\sigma_y(\tau)$ for white PM, flicker PM, and white FM noise processes (see Lesage and Audoin [B63], Howe, Allan, Barnes [B43], Stein [B89], Walter [B102], and Weiss et al. [B104]). As shown in Table E.2, *overlapped* estimates improve the confidence intervals for specific values of M and m. Although $\sigma_y(\tau)$ usually provides a smaller percentage confidence interval than *Mod* $\sigma_y(\tau)$ (see Table E.2 for white PM and flicker PM), the absolute confidence intervals are approximately similar. The reason is that *Mod* $\sigma_y(\tau)$ is typically much smaller than $\sigma_y(\tau)$ for white and flicker PM noise processes.

Table E.2—Confidence intervals for $\sigma_y(\tau)$ (no overlap and full overlap) and *Mod* $\sigma_y(\tau)$ [a]

N = 1025	No overlap ± for 68% $\sigma_y(\tau)$ white PM	Full overlap – for 68% $\sigma_y(\tau)$ white PM	Full overlap + for 68% $\sigma_y(\tau)$ white PM	Full overlap – for 68% *Mod* $\sigma_y(\tau)$ white PM	Full overlap + for 68% *Mod* $\sigma_y(\tau)$ white PM
m = 2	4.4%	2.9%	3.2%	3.1%	3.4%
m = 8	8.7%	2.9%	3.2%	5.2%	6.1%
m = 32	17.4%	3.0%	3.4%	9.7%	14%
m = 128	34.9%	3.1%	3.6%	18%	41%
N = 1025	**Flicker PM**	**Flicker PM**	**Flicker PM**	**Flicker PM**	**Flicker PM**
m = 2	4.4%	2.9%	3.1%	3.0%	3.3%
m = 8	8.7%	3.6%	4.0%	5.7%	6.8%
m = 32	17.4%	5.2%	6.1%	11%	16%
m = 128	34.9%	8.4%	11%	20%	50%
N = 1025	**White FM**	**White FM**	**White FM**	**White FM**	**White FM**
m = 2	3.8%	2.8%	3.0%	3.0%	3.2%
m = 8	7.7%	4.8%	5.6%	5.8%	7.0%
m = 32	15.3%	8.8%	12%	11%	16%
m = 128	30.6%	16%	32%	20%	51%

[a] See Lesage and Audoin [B63], Howe, Allan, Barnes [B43], Stein [B89], Walter [B102], and Weiss et al. [B104].

The confidence limits for frequency domain measures (spectral densities) can be approximated by

$$1 \pm \frac{\mathrm{k}}{\sqrt{N\beta}} \tag{E.7}$$

where

N is the number of averages
β = 1 for FFT spectrum analyzers and β = (resolution BW)/(video BW) for swept spectrum analyzers
k = 1 for 1σ or 68% confidence and k = 2 for 2σ or 95% confidence (see Walls, Percival, Ireland [B100]).

Annex F

(informative)

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